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# The Hourglass Reduction Factor for Asymmetric Colliders*

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**Abstract**

We calculate the geometrical effect of a nonzero bunch length on the luminosity and the beam-beam parameters in the general asymmetric case. We assume that the bunches are three-dimensional gaussian colliding head-on and on-axis; however, the collision point may be displaced longitudinally from the nominal IP. The four beta-functions and the six beam sizes are allowed to have arbitrary values. With these formulas we attempt a limited analytic understanding of the multiparticle tracking simulations that have been carried out for the proposed SLAC/LBL/LLNL B factory [1] when parasitic crossings are ignored. We discuss the electromagnetic pinching effect only qualitatively. We conclude the following: (a) the asymmetric formulas reduce smoothly to the symmetric ones and there is no significant qualitative difference between the two cases; (b) the geometrical reduction in luminosity is $\sim 6\%$ relative to the zero-bunch-length (nominal) value and it is probably compensated (or overcompensated) by the pinching effect; (c) only the vertical beam-beam parameter of the LER is significantly altered by the hourglass effect: the geometrical enhancement of the central positron's $\xi_y$ is $\sim 10\%$ relative to the nominal value, and it is probably made larger by the pinching effect; and (d) the vertical beam-beam parameters of the positrons at the head or tail of the bunch can have instantaneous values much larger than nominal.

## 1 Introduction.

The desire to study the B meson system in detail has led several laboratories to propose designs for asymmetric high-luminosity $e^+e^-$ colliders [1]. Typically the vertical beta-functions are of order $\gtrsim 1$ cm, which is comparable to the bunch lengths. Because the beta-function modulates significantly the transverse bunch size over the effective region of the collision, the luminosity is degraded relative to the zero-bunch-length limit. This modulation also enhances or decreases the beam-beam parameters. This is the so-called "hourglass effect," which is purely geometrical in nature.

All the calculations presented here are strictly static in the sense that no dynamical variation of any sort is taken into account. In particular, the emittances (hence the transverse beam sizes $\sigma^*_{x\pm}$ and $\sigma^*_{y\pm}$ at the IP) are taken as a given input. Similarly, the longitudinal displacement $z$ of a given particle from the center of its own bunch is assumed independent of time, as are the displacements $\Delta_\pm$ of the bunch centers from the nominal IP at the time of collision (in practice, or in realistic multiparticle tracking simulations, the beam sizes normally do reach a constant size, but $z$ oscillates sinusoidally due to synchrotron motion, and the $\Delta_\pm$ may oscillate and/or may jitter).

The proposed designs invoke to a greater or lesser degree a "transparency condition" by virtue of which the beam sizes are pairwise equal [2]. Because of the beam-beam interaction, however, the transparency symmetry is inevitably broken, and the beams become different in size at least to some degree. Expressions available in the literature [3–5] for the hourglass factors for the luminosity and beam-beam parameters are applicable to single-ring colliders and therefore assume some sort of relationship between the beam parameters, such as $\sigma_{x+} = \sigma_{x-}$, $\sigma_{y+} = \sigma_{y-}$, $\sigma_{z+} = \sigma_{z-}$, $\beta^*_{y+} = \beta^*_{y-}$ and $\beta^*_{x+} = \beta^*_{x-} \gg \beta^*_{y+}$ (the suffixes $+$ and $-$ refer to the $e^+$ and $e^-$ beams, respectively, and the superscript $*$ refers to the interaction point). In this

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note we provide generalizations that do not assume any of these relationships, and are, therefore, applicable to the most general asymmetric case. The hourglass factors are expressed in terms of one-dimensional integrals that are easy to evaluate. In the case when $\beta^*_{x+}$ and $\beta^*_{x-}$ are much greater than their vertical counterparts, the expressions are given in terms of modified Bessel functions. In the case of transparent round beams, they are given in terms of the complementary error function.

The multiparticle tracking simulations that have been carried out for the proposed SLAC/LBL/LLNL B factory [1] are "strong-strong" in nature, in which both beams evolve dynamically under their mutual influence. The simulated bunches are three-dimensional, so that nonzero bunch length effects (both geometrical and dynamical) are indeed included. These simulations show that the positron (LER) beam tends to blow up in the vertical direction, and slightly contract or expand in the horizontal direction. The electron (HER) beam tends to slightly contract or expand in both directions. Valuable as they are, multiparticle simulations do not provide much insight into the origin of this dynamics. We attempt here a modest and qualitative analytic understanding of these behaviors for the case in which parasitic collisions are ignored. We calculate a 5.5% geometrical reduction in luminosity and a 9.3% geometrical enhancement of the central positron's vertical beam-beam parameter relative to the zero-bunch-length limits.

In addition to the hourglass effect there is also the "pinching," or "disruption" effect caused by the electromagnetic fields during the collision [6, 7]. We present qualitative arguments that this pinching effect is probably small and favorable for the luminosity, in that it tends to overcome the geometrical reduction. This is consistent with the tracking simulation results in the case when there is no significant LER vertical blowup. However, we show that positrons at the head or tail of the bunch have vertical beam-beam parameters significantly larger than nominal. This is in contrast with the positron's horizontal beam-beam parameter, and both vertical and horizontal parameters for the electrons, all of which remain relatively unchanged from nominal for most particles. The pinching effect is possibly detrimental for these particles.

It is probably interesting that the enhancement of the positron's vertical beam-beam parameter, as shown in this note, seems correlated with the positron's vertical blowup seen in the simulations. We do not attempt here to pin down the possible connection. Because the primary objective of the simulations has so far been the study of the dynamics of the *beam core*, all macroparticles have been confined longitudinally to the region $|z| < 2\sigma_z$ around the bunch center. For larger values of $|z|$ the relatively large value of the vertical beam-beam parameter, as shown below, implies a potentially significant effect on the dynamics of the bunch tails. This would probably affect the beam lifetime more than the short-term average luminosity.

## 2 Expression for the luminosity.

Consider two bunches, one of electrons and one of positrons, whose particle densities are $\rho_-$ and $\rho_+$, respectively. We assume that they are normalized to the number of particles in each bunch,

$$\int dxdyds\, \rho_\pm(x, y, s) = N_\pm \tag{2.1}$$

If these two bunches move in equal and opposite directions with speeds $v_+$ and $v_-$ then the luminosity for the collision is given by

$$\mathcal{L} = (v_+ + v_-) \int dtdxdyds\, \rho_+(x, y, s - v_+t)\rho_-(x, y, s + v_-t) \tag{2.2}$$

where the integral extends from $-\infty$ to $+\infty$ for all four variables of integration. Note that this expression has dimensions of length$^{-2}$ rather than the usual length$^{-2}$time$^{-1}$. This is because (2.2) is the luminosity *per bunch collision*, not per unit time. The usual expression for the luminosity applicable to the periodic collision of like bunches at a given interaction point is obtained by multiplying (2.2) by the bunch collision frequency.

The above expression is valid for arbitrary bunch densities, as long as the two bunches are moving along in equal but opposite directions. In particular, if the densities have some symmetry axis defining a center in the transverse plane, as in the usual case of bunches of elliptical cross sections, Eq. (2.2) is valid even if the bunches collide off-axis.

We now specialize to the case of tri-gaussian bunches moving towards each other with speed $c$,

$$\rho_\pm(x,y,s\mp ct) = \frac{N_\pm \exp\left(-\dfrac{x^2}{2\sigma_{x\pm}^2} - \dfrac{y^2}{2\sigma_{y\pm}^2} - \dfrac{(s\mp ct)^2}{2\sigma_{z\pm}^2}\right)}{(2\pi)^{3/2}\sigma_{x\pm}\sigma_{y\pm}\sigma_{z\pm}} \tag{2.3}$$

Since the bunch centers obey $s = \pm ct$, the time $t = 0$ is the central collision time. We assume that the transverse rms bunch sizes $\sigma_{x\pm}$ and $\sigma_{y\pm}$ depend on the path length $s$, but that the longitudinal rms sizes $\sigma_{z\pm}$ are constant. Then (2.2) yields

$$\mathcal{L} = \frac{N_+N_-}{\pi\sqrt{2\pi}\sqrt{\sigma_{z+}^2+\sigma_{z-}^2}} \int_{-\infty}^{\infty} ds\, \frac{\exp\left(-\dfrac{2s^2}{\sigma_{z+}^2+\sigma_{z-}^2}\right)}{\sqrt{(\sigma_{x+}^2+\sigma_{x-}^2)(\sigma_{y+}^2+\sigma_{y-}^2)}} \tag{2.4}$$

In the special case in which the transverse rms bunch sizes are constant or do not vary significantly over a distance $(\sigma_{z+}^2+\sigma_{z-}^2)^{1/2}$, we obtain from (2.4) the familiar short-bunch expression for the luminosity,

$$\mathcal{L} = \mathcal{L}_0 \equiv \frac{N_+N_-}{2\pi\sqrt{(\sigma_{x+}^{*2}+\sigma_{x-}^{*2})(\sigma_{y+}^{*2}+\sigma_{y-}^{*2})}} \tag{2.5}$$

where the superscript $*$ refers to the collision point, $s = 0$.

## 2.1 Luminosity reduction factor for drifts.

Let us assume now that the bunches collide in a drift section with zero dispersion, and that the interaction point is a symmetry point of the lattice. We assume also that the bunches collide at this point with no longitudinal displacement. Then the rms bunch sizes vary away from the collision point according to

$$\sigma_{x\pm}^2 = \sigma_{x\pm}^{*2} \times \left(1 + \frac{s^2}{\beta_{x\pm}^{*2}}\right) \tag{2.6}$$

with a corresponding expression for $\sigma_{y\pm}^2$. From (2.4) we obtain, after a straighforward scaling of the integration variable $s$ and factoring out the nominal expression (2.5), the reduction factor

$$R(u_x,u_y) \equiv \frac{\mathcal{L}}{\mathcal{L}_0} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{\exp(-u^2)}{\sqrt{(1+u^2/u_x^2)(1+u^2/u_y^2)}} \tag{2.7}$$

where $u$ is a dummy integration variable proportional to $s$, and $u_x$ is defined by

$$u_x^2 = \frac{2(\sigma_{x+}^{*2}+\sigma_{x-}^{*2})}{(\sigma_{z+}^2+\sigma_{z-}^2)\left(\sigma_{x+}^{*2}/\beta_{x+}^{*2}+\sigma_{x-}^{*2}/\beta_{x-}^{*2}\right)} \tag{2.8}$$

with a corresponding expression for $u_y$. Eqs. (2.7) and (2.8) constitute the basic result of this section. We exhibit $R(u_x,u_y)$ plotted vs. $u_y$ for selected values of $u_x$ in Fig. 1. The integral was performed with Simpson's algorithm (program `HOURGLASS_LUM.F`).

The parameters $u_x$ and $u_y$ measure the relative sizes of the beta-functions to beam lengths. For the important transparent case in which the beam sizes are pairwise equal, i.e., $\sigma_{x+}^* = \sigma_{x-}^*$ and $\sigma_{y+}^* = \sigma_{y-}^*$, $u_x$ and $u_y$ lose their explicit dependence on transverse beam size. If, in addition, $\sigma_{z+} = \sigma_{z-}$, then these parameters reduce to

$$\left.\begin{aligned} u_x^2 &= \frac{2\beta_{x+}^{*2}\beta_{x-}^{*2}}{\sigma_z^2\left(\beta_{x+}^{*2}+\beta_{x-}^{*2}\right)} \\ u_y^2 &= \frac{2\beta_{y+}^{*2}\beta_{y-}^{*2}}{\sigma_z^2\left(\beta_{y+}^{*2}+\beta_{y-}^{*2}\right)} \end{aligned}\right\} \quad \text{if } \sigma_{x+}=\sigma_{x-},\ \sigma_{y+}=\sigma_{y-},\ \text{and } \sigma_{z+}=\sigma_{z-} \tag{2.9}$$

### 2.2 Special cases for the luminosity reduction factor.

In some limiting cases one can find useful analytic expressions for $R(u_x, u_y)$. Firstly, when both bunches have zero length, one recovers the correct result from (2.7) by setting $u_x = u_y = \infty$,

$$R(\infty, \infty) = 1 \tag{2.10}$$

while it is easy to see that $R < 1$ for all other cases. In the limit when both $u_x$ and $u_y$ are large, we obtain a first correction to this asymptotic limit,

$$R(u_x, u_y) = 1 - \frac{1}{4}\left(\frac{1}{u_x^2} + \frac{1}{u_y^2}\right) + \cdots \tag{2.11}$$

Of special interest is the case when both beams are flat, with bunch lengths comparable to the vertical beta functions at the IP and horizontal sizes much larger than the vertical. In this case $u_x \gg 1$ and we obtain, upon making the change of variables $u \to u_y \sinh(u/2)$ and setting $u_x = \infty$,

$$R(\infty, u_y) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{e^{-u^2}}{\sqrt{1 + u^2/u_y^2}} = \frac{u_y}{\sqrt{\pi}}\, e^{u_y^2/2} K_0(u_y^2/2) \tag{2.12}$$

where $K_0$ is a Bessel function. This result is of similar form as Ivanov et al.'s [4], but is of more general validity because it does not assume $\beta^*_{y+} = \beta^*_{y-}$ or $\sigma_{z+} = \sigma_{z-}$ or $\sigma^*_{y+} = \sigma^*_{y-}$. In the limiting cases $u_y \to 0$ and $u_y \to \infty$ we obtain

$$R(\infty, u_y) = \begin{cases} -\dfrac{u_y}{\sqrt{\pi}}\left(1 + \frac{1}{2}u_y^2\right)\left(\gamma + \log(u_y^2/4)\right) + \mathcal{O}(u_y^4), & u_y \to 0 \\ 1 - \dfrac{1}{4u_y^2} + \mathcal{O}(u_y^{-4}), & u_y \to \infty \end{cases} \tag{2.13}$$

where $\gamma$ is here Euler's constant. Fig. 2 depicts Eq. (2.12) along with these two asymptotic expressions.

Also of interest is the case when $u_x = u_y$, which includes the transparent round-beam case. In this case the luminosity reduction factor (2.7) becomes

$$R(u_x, u_x) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{e^{-u^2}}{1 + u^2/u_x^2} = \sqrt{\pi}\, u_x\, e^{u_x^2} \mathrm{erfc}(u_x) \tag{2.14}$$

where $\mathrm{erfc}(x)$ is the complementary error function [8]. In the limiting cases $u_x \to 0$ and $u_x \to \infty$ we obtain

$$R(u_x, u_x) = \begin{cases} \sqrt{\pi}\, u_x \left(1 - \dfrac{u_x}{\sqrt{\pi}} + u_x^2\right) + \mathcal{O}(u_x^4), & u_x \to 0 \\ 1 - \dfrac{1}{2u_x^2} + \mathcal{O}(u_x^{-4}), & u_x \to \infty \end{cases} \tag{2.15}$$

Eq. (2.14) is plotted in Fig. 3 along with these limiting forms.

## 3 The hourglass effect for the beam-beam parameters.

In this section we calculate, in first order in perturbation theory, the effect that the modulation of the beta-function has on the nominal beam-beam parameters $\xi$. As before, we consider only relativistic, upright, tri-gaussian bunches colliding head-on and on-axis with no longitudinal displacement from the nominal collision point, i.e., $\Delta_+ = \Delta_- = 0$ (see Fig. 4).

We consider only the "worst case," corresponding to particles infinitesimally close to the axis of one bunch passing through the opposing bunch. These are the particles that experience the largest beam-beam effect. The integrated strength of the beam-beam kick of the particle at the center of the bunch is what is usually referred to as the beam-beam parameter $\xi$ of that beam (there are, of course, four beam-beam parameters altogether, two for each beam). Because we calculate $\xi$ only to first order, we assume that the particle in question follows a straight line trajectory with constant speed $c$.

To fix ideas, we assume that the positrons move to the right and the electrons to the left with distributions given by (2.3). We focus our attention on a single particle, say a positron, as it passes through the opposing electron bunch. This particle has $x \simeq y \simeq 0$ and is displaced longitudinally by a finite distance $z$ from the center of its own bunch, as sketched in Fig. 4. We assume that $z$ does not vary during the collision process, which is a very good approximation in practice. In the relativistic limit it is easy to show from Maxwell's equations that the electric and magnetic field in the lab frame produced by the electrons, with distribution (2.3), are given by

$$\mathbf{E}_- = \frac{-2eN_- \left(\mathbf{i}\, x/\sigma_{x-} + \mathbf{j}\, y/\sigma_{y-}\right)}{\sqrt{2\pi}\,\sigma_{z-}\,(\sigma_{x-} + \sigma_{y-})} \exp\left(-(s+ct)^2/2\sigma_{z-}^2\right) \tag{3.1a}$$

$$\mathbf{B}_- = \frac{\mathbf{v}_-}{c} \times \mathbf{E}_- \tag{3.1b}$$

where the electron charge is taken to be $-e$. The positron in question obeys $s = ct + z$, and the force it experiences is, in the relativistic limit,

$$\mathbf{F}_+ = e\left(\mathbf{E}_- + \frac{\mathbf{v}_+}{c} \times \mathbf{B}_-\right) = 2e\mathbf{E}_-|_{ct=s-z} \tag{3.2}$$

As a result of this beam-beam collision the vertical focusing function of the positron is shifted by an amount

$$\Delta K_{y+}(s) = -\frac{F_{y+}/y}{mc^2\gamma_+} \tag{3.3}$$

with a corresponding expression for the horizontal counterpart. Here $mc^2\gamma_+$ is the positron energy. The corresponding integrated beam-beam kick strength is, in first order,

$$\xi_{y+}(z) = \frac{1}{4\pi}\int ds\, \beta_{y+}(s)\Delta K_{y+}(s) = \frac{r_0 N_-}{\pi\sqrt{2\pi}\gamma_+\sigma_{z-}}\int ds\, \frac{\beta_{y+}(s)\, e^{-(2s-z)^2/2\sigma_{z-}^2}}{\sigma_{y-}(\sigma_{x-} + \sigma_{y-})} \tag{3.4}$$

where $r_0$ is the classical electron radius, $r_0 = e^2/mc^2$. The expressions for the remaining three beam-beam parameters $\xi_{x+}$, $\xi_{x-}$ and $\xi_{y-}$ are obtained from (3.4) by the obvious substitutions. Eq. (3.4) is the basic result of this section.[1]

Because of this beam-beam interaction, the positron's tune $\nu$ is shifted by an amount $\Delta\nu$ which is found, in first order in $\xi$, from the equation $\cos(2\pi(\nu + \Delta\nu)) = \cos 2\pi\nu - 2\pi\xi \sin 2\pi\nu$ (four separate equations apply for each of the four tune shifts). If $\nu$ is not too close to an integer or half-integer, one finds the familiar result $\Delta\nu \simeq \xi$. If the bunch length $\sigma_{z-}$ is much shorter than the characteristic length of variation of the transverse sizes and the beta-function, Eq. (3.4) yields the familiar result for the particle at the center of the bunch ($z = 0$)

$$\xi_{0y+} = \frac{r_0 N_- \beta^*_{y+}}{2\pi\gamma_+\sigma^*_{y-}(\sigma^*_{x-} + \sigma^*_{y-})} \tag{3.5}$$

## 3.1 The beam-beam aggravating factor for drift sections.

We now specialize the calculation to the case of a dispersionless drift section, as appropriate to an interaction region. The beta-functions and the square of the transverse beam sizes vary with $s$ according to Eq. (2.6). By a straightforward rescaling[2] of the variable $s$ and factoring out the zero-bunch length expression (3.5),

[1] For Eq. (3.4) to be applicable as written to the electron beam, one must adopt the convention that $z > 0$ represents the *tail*, not the head, of the e$^-$ bunch.

[2] The rescaling used here is different from the one used in the luminosity reduction factor; however, we still call $u$ the resulting dummy integration variable.

we arrive at a result called the "aggravating factor," [3],

$$R_{y+}(z) \equiv \frac{\xi_{y+}(z)}{\xi_{0y+}} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{(1+u^2/u_1^2)\,\exp(-(u-u_0)^2)}{\sqrt{1+u^2/u_2^2}\left(v\sqrt{1+u^2/u_2^2}+h\sqrt{1+u^2/u_3^2}\right)} \tag{3.6}$$

where

$$\begin{gathered} h = \frac{\sigma_{x-}^*}{\sigma_{x-}^*+\sigma_{y-}^*}, \quad v = \frac{\sigma_{y-}^*}{\sigma_{x-}^*+\sigma_{y-}^*} \\ u_0 = \frac{z}{\sqrt{2}\sigma_{z-}}, \quad u_1 = \frac{\sqrt{2}\beta_{y+}^*}{\sigma_{z-}}, \quad u_2 = \frac{\sqrt{2}\beta_{y-}^*}{\sigma_{z-}}, \quad u_3 = \frac{\sqrt{2}\beta_{x-}^*}{\sigma_{z-}} \end{gathered} \tag{3.7}$$

It should be noted that, depending on the relative size of the beta-functions and beam sizes at the interaction point, the beam-beam enhancement factors can be $> 1$ or $< 1$, as opposed to the luminosity reduction factor, which is always $< 1$.

The expressions for $R_{x+}$, $R_{x-}$ and $R_{y-}$ are obtained from (3.6) by the replacements $x \leftrightarrow y$ and/or $+ \leftrightarrow -$ in $h$, $v$ and the $u_i$'s.

## 3.2 Special cases for the aggravating factor at $z = 0$.

We now consider only the particle at the center of the bunch ($z = 0$).

### 3.2.1 Pairwise-equal $\beta^*$'s.

The first special case we consider arises when $\beta_{x+}^* = \beta_{x-}^*$ and $\beta_{y+}^* = \beta_{y-}^*$, which is a condition that might be chosen as part of a specific transparent-symmetric design.[3] In this case $u_1 = u_2$ for both $x$ and $y$, and the enhancement factor becomes

$$R_{y+}(0) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{\sqrt{1+u^2/u_2^2}\,\exp(-u^2)}{v\sqrt{1+u^2/u_2^2}+h\sqrt{1+u^2/u_3^2}} \tag{3.8}$$

while $R_{x+}(0)$ is obtained, in this case, by replacing $u_2 \to u_3$ in the numerator and leaving the denominator unchanged. It is easy to see, then, that

$$hR_{x+}(0) + vR_{y+}(0) = 1 \tag{3.9}$$

Given that $h + v = 1$, this implies that either $R_{x+}(0) > 1$, or $R_{y+}(0) > 1$, or $R_{x+}(0) = R_{y+}(0) = 1$.

### 3.2.2 Flat beams.

The second special case we consider is that of flat beams. We assume that $\sigma_{x+} \gg \sigma_{y+}$, $\sigma_{x-} \gg \sigma_{y-}$, $\beta_{x+}^* \gg \beta_{y+}^*$ and $\beta_{x-}^* \gg \beta_{y-}^*$. In this case we find

$$R_{x+}(0) \simeq R_{x-}(0) \simeq 1 \tag{3.10}$$

and

$$R_{y+}(0) \simeq \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{(1+u^2/u_1^2)\,\exp(-u^2)}{\sqrt{1+u^2/u_2^2}} = \frac{u_2}{2\sqrt{\pi}}\, e^{u_2^2/2} \left[(2-\rho)K_0(u_2^2/2) + \rho K_1(u_2^2/2)\right] \tag{3.11}$$

where

$$\rho \equiv \left(\frac{u_2}{u_1}\right)^2 = \left(\frac{\beta_{y-}^*}{\beta_{y+}^*}\right)^2 \tag{3.12}$$

and $K_0$, $K_1$ are Bessel functions. In this special case the expression for $R_{y-}(0)$ is obtained from (3.11) by the exchange $u_1 \leftrightarrow u_2$.

[3] This equality of the beta-bunctions is not required, however, by the usual dictates of transparency symmetry; see Ref. 2.

### 3.2.3 $\beta_x = \beta_y$.

A third special case arises when $u_2 = u_3$ for both beams. This implies that $\beta^*_{x+} = \beta^*_{y+}$ and $\beta^*_{x-} = \beta^*_{y-}$. These equalities might be chosen, for example, as part of a round-beam design. We allow, however, for the possibility that $\beta^*_{x+} \neq \beta^*_{x-}$ and we assume nothing about the emittances, hence the six rms beam sizes $\sigma$ might all be different. Since $h+v=1$, the expression for $R_{y+}(0)$, Eq. (3.6), becomes, with no approximation,

$$R_{x+}(0) = R_{y+}(0) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{1+u^2/u_1^2}{1+u^2/u_2^2} \exp(-u^2) = \rho + (1-\rho)\sqrt{\pi}\, u_2\, e^{u_2^2} \operatorname{erfc}(u_2) \tag{3.13}$$

where $\rho$ is the same as above, Eq. (3.12). The expression for $R_{x-}(0)$ (which is equal to $R_{y-}(0)$) is obtained from (3.13) by the exchange $u_1 \leftrightarrow u_2$.

Finally we note, as a special case of the above, that if all four beta-functions are equal, $\beta^*_{x+} = \beta^*_{y+} = \beta^*_{x-} = \beta^*_{y-}$, then $u_1 = u_2 = u_3$ and we obtain

$$R_{x+}(0) = R_{y+}(0) = R_{x-}(0) = R_{y-}(0) = 1 \tag{3.14}$$

*regardless of the transverse or longitudinal beam sizes.* This result should be obvious from Eq. (3.4); it is noteworthy because it allows the possibility to eliminate the hourglass effect on the beam-beam parameters, should this be considered a problem (however, this possibility seems natural only for a round-beam design, such as for a proton collider).

## 3.3 The aggravating factor for $z \neq 0$.

For particles away from the center of the bunch, which have $z \neq 0$, the aggravating factor does not seem to be expressible in terms of special functions in the special cases considered above. However, three useful properties can be easily seen from Eq. (3.6). The first property is the symmetry

$$R_{y+}(z) = R_{y+}(-z), \quad \text{or} \quad \xi_{y+}(z) = \xi_{y+}(-z) \tag{3.15}$$

which is also true of the remaining three aggravating factors (or beam-beam parameters). Physically, this means that the particles at the head of the bunch suffer the same beam-beam tune shift as the particles at the tail of the bunch, provided they are at the same distance from the center. This result can be understood from time-reversal symmetry, which is a valid symmetry on account of the assumed beta-function symmetry about the interaction point, and the assumed lack of bunch distortion. Thus, if one imagines the collision process run backwards in time, a particle at the head of the bunch becomes a particle at the tail of the bunch. Time-reversal symmetry implies, therefore, that these two particles experience exactly the same forces, and hence exactly the same beam-beam tune shifts.

A second mathematical property of the aggravating factor is that it saturates to a limit when $z \to \infty$. The limit is easily shown to be

$$\lim_{z\to\infty} R_{y+}(z) = \frac{u_2^2 u_3}{u_1^2(vu_3 + hu_2)} \tag{3.16}$$

This limit is sensibly reached when $u_0 \gg u_1, u_2, u_3$ corresponding to particles with $|z| \gg \beta^*$, where $\beta^*$ represents here the largest of the four beta-functions at the interaction point, hence this limit is not applicable to any of the existing B factory designs. The expressions for the remaining three factors are obtained by the obvious substitutions on both sides of (3.16).

Finally we note that, just as in the $z = 0$ case, if all four beta-functions are equal, we obtain

$$R_{x+}(z) = R_{y+}(z) = R_{x-}(z) = R_{y-}(z) = 1 \tag{3.17}$$

for all values of $z$. This follows straightforwardly from Eq. (3.4).

Table 1: Nominal parameters for APIARY 6.3–D.

| | LER ($e^+$) | HER ($e^-$) |
|---|---|---|
| $\sigma_x^*$ [$\mu$m] | 186 | 186 |
| $\sigma_y^*$ [$\mu$m] | 7.35 | 7.35 |
| $\sigma_z$ [cm] | 1 | 1 |
| $\beta_x^*$ [cm] | 37.5 | 75 |
| $\beta_y^*$ [cm] | 1.5 | 3 |
| $\xi_{0x}$ | 0.03 | 0.03 |
| $\xi_{0y}$ | 0.03 | 0.03 |

# 4 Numerical application to the SLAC/LBL/LLNL B factory.

From the proposed SLAC/LBL/LLNL B factory conceptual design report [1] we assume parameter values as listed in Table 1.

From Eq. (2.8) we obtain $u_x = 47.43$ and $u_y = 1.897$. Because $u_x$ is large enough, the approximate expression (2.12) is valid with an estimated accuracy better than 1 part in 1,000. We obtain, either from (2.7) or (2.12) [8],

$$R = 0.945 \tag{4.1}$$

which implies a 5.5% reduction in luminosity relative to the zero-bunch-length calculation.

For the beam-beam aggravating factors for the particle at the bunch center we obtain, from (3.6),

$$\begin{array}{ll} R_{y+}(0) = 1.093, & R_{y-}(0) = 0.977 \\ R_{x+}(0) = 1.000, & R_{x-}(0) = 0.998 \end{array} \tag{4.2}$$

which shows, in particular, an enhancement of 9.3% for the nominal vertical beam-beam parameter of the positron beam.

For particles away from the center of the bunch, $R_{y+}$ grows fairly quickly as the distance increases. Fig. 5 shows $\xi_{y+}$ as a function of the positron's distance away from the center of the bunch. $\xi_{y+}$ grows to $\sim 0.1$ for a 6-$\sigma_z$ particle, and reaches 0.18 at $z = 10\sigma_z$. The remaining three aggravating factors deviate significantly from unity only for $z > 10\sigma_z$. Fig. 6 shows all four aggravating factors as a function of the particle distance away from the center of the bunch. We have deliberately extended the horizontal scale up to an unphysical value of $z = 1000\sigma_z = 10$ m in order to show the saturation effect (3.16). Of course, when the beam-beam parameters become large, the first-order approximation used in this note breaks down and the results become unreliable.

The physical interpretation for the large values of the vertical beam-beam parameter of the positrons at the head or tail of the bunch is that these particles sample, on the average, a much larger vertical beta-function than those at the bunch center. This is because the LER vertical beta-function is quite small at the interaction point, $\beta_{y+}^* = 1.5$ cm, so that it grows quickly away from this point. The almost linear increase of $\xi_{y+}$ for $z \gtrsim 5\sigma_{z+}$, which is seen in Fig. 5, can also be easily understood from from Eq. (3.4). The integrand is effectively concentrated in the neighborhood of $s = z/2$ with a width of a few $\sigma_{z-}$. In the region $z \gtrsim 5\sigma_{z+}$ the dominant variation of the integrand comes from $\beta_{y+}^*$ and $\beta_{y-}^*$ so that

$$\xi_{y+}(z) \propto \int ds \frac{\beta_{y+}(s)}{\sqrt{\beta_{y-}(s)}}\, e^{-(2s-z)^2/2\sigma_{z-}^2} \simeq \frac{\beta_{y+}(z/2)}{\sqrt{\beta_{y-}(z/2)}} \tag{4.3}$$

where we have used the method of steepest descent, which is quite a reasonable approximation in this parameter regime. The numerical factors can be more easily collected from Eq. (3.6). The corresponding calculation yields

$$\xi_{y+}(z) \simeq \xi_{0y+} \times \frac{1 + u_0^2/u_1^2}{\sqrt{1 + u_0^2/u_2^2}}$$

$$
\begin{aligned}
&\simeq \xi_{0y+} \times \frac{u_0 u_2}{u_1^2} \\
&= \xi_{0y+} \times \frac{z\beta^*_{y-}}{2\beta^{*2}_{y+}} \\
&= 0.02\, z \ [\mathrm{cm}]
\end{aligned}
\tag{4.4}
$$

where we have used the definitions (3.7) and have substituted nominal values from Table 1. The numerical factor of 0.02 turns out to be in very good agreement with what is inferred from the exact result plotted in Fig. 5.

# 5 Off–IP collisions.

Because of possible RF jitter or phasing errors, the bunches may not collide exactly at the nominal IP (defined by the common waist of the beta-functions). Thus we assume that the bunch centers are displaced by amounts $\Delta_\pm$ from their ideal values (see Fig. 4),

$$
\begin{aligned}
&e^+ \text{ bunch center:} \quad s_+ = ct + \Delta_+ \\
&e^- \text{ bunch center:} \quad s_- = -ct + \Delta_-
\end{aligned}
\tag{5.1}
$$

where $s = 0$ defines the location of the nominal IP. The actual collision point, which we call $s_c$, is found by setting $s_+ = s_-$, hence

$$
s_c = \tfrac{1}{2}(\Delta_+ + \Delta_-) \tag{5.2}
$$

with the implicit convention that $s_c > 0$ means downstream of the IP in the direction of the e$^+$ beam.

The luminosity formula, Eq. (2.2), is replaced by (for $v_+ = v_- = c$),

$$
\mathcal{L} = 2c \int dtdxdyds\, \rho_+(x, y, s - ct - \Delta_+)\rho_-(x, y, s + ct - \Delta_-) \tag{5.3}
$$

Manipulations similar to the previous ones yield the reduction factor

$$
R(u_x, u_y, u_c) = \int\limits_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \frac{\exp(-(u-u_c)^2)}{\sqrt{(1+u^2/u_x^2)(1+u^2/u_y^2)}} \tag{5.4}
$$

where

$$
u_c^2 = \frac{2s_c^2}{\sigma_{z+}^2 + \sigma_{z-}^2} \tag{5.5}
$$

and $u_x,\ u_y$ are the same as before, Eq. (2.8). The ideal case (collision exactly at the IP) corresponds to $u_c = 0$. A few properties of the reduction factor are:

$$
R(u_x, u_y, u_c) = R(u_x, u_y, -u_c) \tag{5.6a}
$$

$$
R(u_x, u_y, u_c) < R(u_x, u_y, 0) \quad \text{for } u_c \neq 0 \tag{5.6b}
$$

$$
R(u_x, u_y, u_c) \sim \frac{u_x u_y}{u_c^2} \quad \text{for } u_c \gg u_x,\ u_y \tag{5.6c}
$$

For APIARY 6.3-D parameters, Fig. 7 shows $R(u_x, u_y, u_c)$ plotted vs. $s_c$. It is seen that the reduction in luminosity is quite smooth and not very sensitive to the offset for reasonable values one might expect for $s_c$ (this assumes, of course, that there is no accompanying beam blowup).

The beam-beam parameters are also computed as before. The electric field produced by the electron bunch, Eq. (3.1), is now proportional to $\exp(-(s+ct-\Delta_-)^2/2\sigma_{z-}^2)$, and the force on the positron, Eq. (3.2), is evaluated at $s = ct + z + \Delta_+$. The net result is that the expression for the beam-beam parameter is the same as the one for the ideal (on–IP) case, Eq. (3.4), except that the argument is $z + 2s_c$ instead of $z$,

$$
\xi(z)_{\mathrm{off-IP}} = \xi(z + 2s_c)_{\mathrm{on-IP}} \tag{5.7}
$$

for all four beam-beam parameters. This property is also true, of course, of the aggravating factors. Fig. 8 shows the vertical $\xi$ of a positron displaced longitudinally by $z$ from the center of its own bunch when the collision point is offset by 0 and $\pm 1$ cm from the nominal IP, for APIARY 6.3–D parameters.

# 6 The electromagnetic pinching effect.

The calculations of the luminosity reduction factor and the beam-beam aggravating factors presented in this note are purely geometrical in nature. In other words, the bunch distributions are asssumed unchanged during the collision except for the variation of the beta-functions. In practice, of course, the electromagnetic forces experienced by the particles distort the bunch shape at least to some extent.

In the limit of weak beams, the geometrical results can be thought of as a first order approximation to the full-fledged dynamics. In this section we try to ascertain the magnitude of the pinching effect for the proposed SLAC/LBL/LLNL B factory. We argue that, indeed, the geometrical results are a reasonably good approximation to the full dynamics.

Our qualitative argument is based on results by Chen, and by Chen and Yokoya [7]. These come from multiparticle simulations for single-pass, symmetric, beam collisions. For flat bunches that are uniform in $x$ and gaussian in $y$ and $s$, Chen concludes that the enhancement of the luminosity is represented, to an accuracy of $\pm 10\%$, by the empirical fit

$$H_D \equiv \frac{\mathcal{L}}{\mathcal{L}_0} = \left\{ 1 + D^{1/4} \left( \frac{D^3}{1+D^3} \right) \left[ \ln(\sqrt{D}+1) + 2\ln\left(\frac{0.8}{A}\right) \right] \right\}^{1/3} \tag{6.1}$$

where

$$A = \frac{\sigma_z}{\beta_y^*}, \qquad D = 4\pi\xi_y \frac{\sigma_z}{\beta_y^*} \tag{6.2}$$

Roughly speaking, in this result the parameter $A$ describes geometrical effects while $D$ describes electromagnetic effects. One of the assumptions made in the simulations is that the parameters are such that the purely geometrical effect is small. This assumption is reflected in (6.1): if we set $D = 0$ we obtain $H_D = 1$ for all values of $A$.

Furthermore it should be remembered that these simulations are for single-pass, not repetitive, collisions. Therefore, when applying these results to any circular collider, a potentially important dependence on the tune of the machine may be missed. Nevertheless, the above formula should give an estimate of the relative importance of the electromagnetic effects. Also, the beam symmetry assumed in these simulations does not hold true for the SLAC/LBL/LLNL B factory; we conjecture, however, that, in order to get a qualitative estimate of the effect, we may replace $A$ by its natural generalization, $A \to u_y^{-1} = 0.53$, so that $D = 0.20$. For these values we obtain from (6.1)

$$H_D = 0.998 \tag{6.3}$$

which means that there is essentially no deviation from the nominal value. Therefore, to within 10%, we conclude that the geometrical reduction in luminosity is compensated by the pinching effect. This result is consistent with the multiparticle tracking simulation results for the B factory, which show that there is little deviation from nominal behavior for nominal values of the parameters, if parasitic collisions are ignored [1].

This result may also be interpreted by stating that the pinching effect alters the beta-function at the interaction point, effectively replacing $\beta^*$ by a smaller, "dynamical," beta-function. If this interpretation is correct, it implies that the pinching effect tends to make the the beam-beam aggravating factors larger than those already calculated from the purely geometrical hourglass effect.

# 7 Conclusions.

We have presented the formulas for the hourglass luminosity reduction factor and beam-beam aggravating factors in the general asymmetric case. As in the symmetric case, the reduction factor is, roughly speaking, a sensitive function of $\beta^*/\sigma_z$. Unlike the symmetric case, however, the luminosity reduction factor depends on the transverse bunch sizes in addition to the bunch lengths and beta-functions. In specific instances of flat-beam or round-beam designs, the formulas become expressible in terms of special functions. The asymmetric formulas reduce smoothly to those corresponding to the symmetric case, and there is no qualitative difference between these two cases.

A numerical application to the SLAC/LBL/LLNL B factory shows a 5.5% geometrical reduction of the luminosity and a 9.3% geometrical enhancement of the central positron's $\xi_{y+}$ relative to the nominal values. If the pairwise equality of the transverse beam sizes did not hold, which is a generic consequence of the dynamics, the luminosity reduction would be slightly greater. For example, if the vertical size of the LER beam were twice its nominal value (which is a pessimistic possibility), the luminosity reduction would be 7% instead of 5.5%, while the beam-beam aggravating parameters would remain essentially unaltered (of course, the luminosity would be $\sim 40\%$ smaller than the nominal value, even without consideration of the hourglass effect, simply because of the beam blowup).

The positrons at the head or tail of the bunch have a substantially higher $\xi_{y+}$ than the central positron due to the fact that they sample, on average, a much higher $\beta^*_{y+}$ during the collision process. Positrons with $x \simeq y \simeq 0$ and $|z| > 6\sigma_z$ have $\xi_{y+} > 0.1$; we conjecture that these particles will be quickly lost. It should be remembered, however, that, in reality, synchrotron motion implies that $z$ oscillates about 0 with a period $\sim 25$ turns. Thus any given particle will have a time-averaged $\xi$ much smaller than the above values. In all calculations presented here we have set $z =$constant, so that our conclusions may be unduly pessimistic. Furthermore, particles with $x \neq 0$ or $y \neq 0$ have smaller $\xi$-parameters than those on axis. All multiparticle tracking simulations carried out to this date (8/91) for the SLAC/LBL/LLNL B factory have aimed at the study of the beam core only: in all cases particles have been confined to $|z| \leq 2\sigma_z$. Therefore these simulations are not sensitive to the behavior of large-$|z|$ particles. In any case, Eq. (4.4) shows a simple way to decrease the $\xi$ parameters at the head or tail of the bunch, by appropriately balancing $\beta^*_{y+}$ and $\beta^*_{y-}$.

If the bunches collide at a point longitudinally displaced from the nominal IP, the luminosity is smoothly reduced from the ideal case. The beam-beam parameters at the head or tail of the bunch, however, are sensitive to this offset, and can easily become quite large.

We estimate the electromagnetic pinching effect to be small, since it modifies the results of the geometrical calculations by $\sim \pm 10\%$. It is probably beneficial for the luminosity, and it is probably detrimental for the beam-beam parameters.

For all these reasons one can say that, generally speaking, the hourglass effect has a greater influence on the dynamics of the bunch tails than on the the dynamics of the core, at least for a range of parameters like those of presently proposed B factories. Therefore this effect is more important for the beam lifetime than for the instantaneous luminosity.

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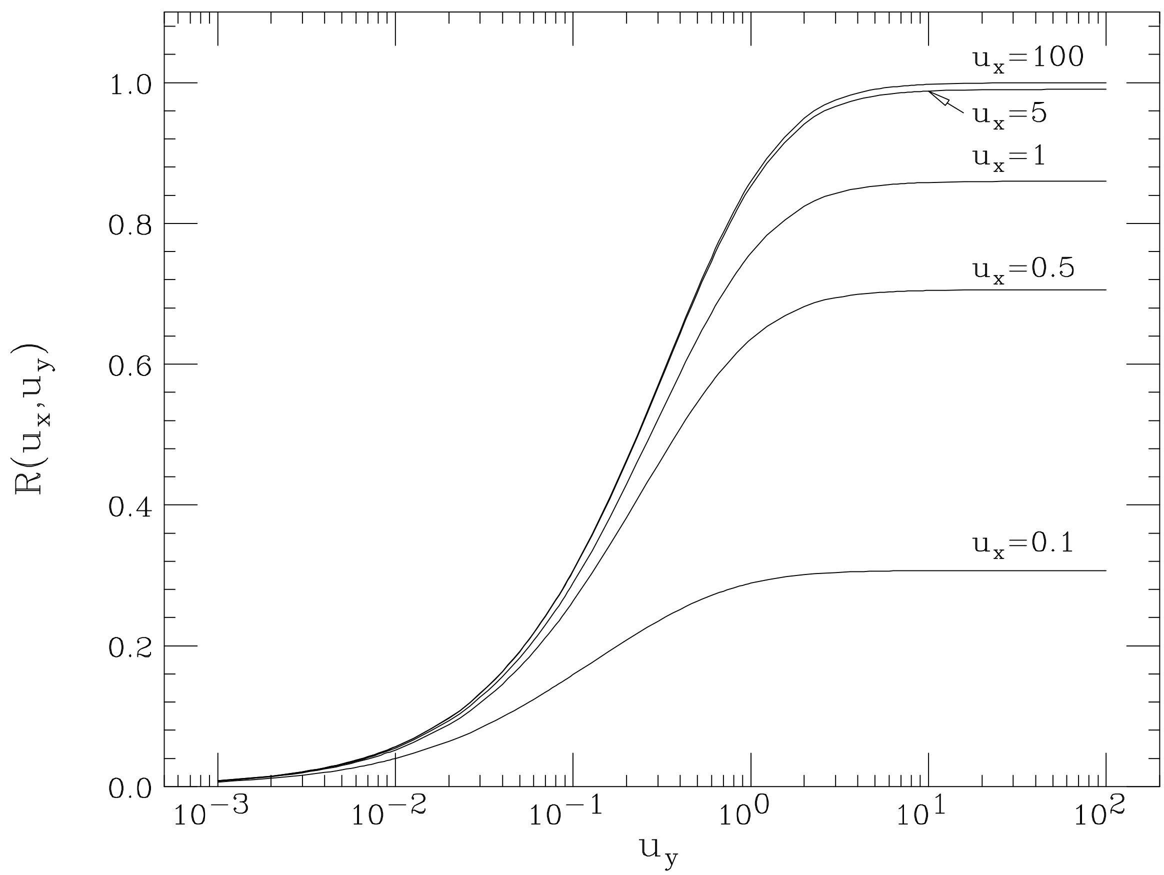


Figure 1: The reduction factor, Eq. (2.7), plotted vs. $u_y$ for various values of $u_x$.

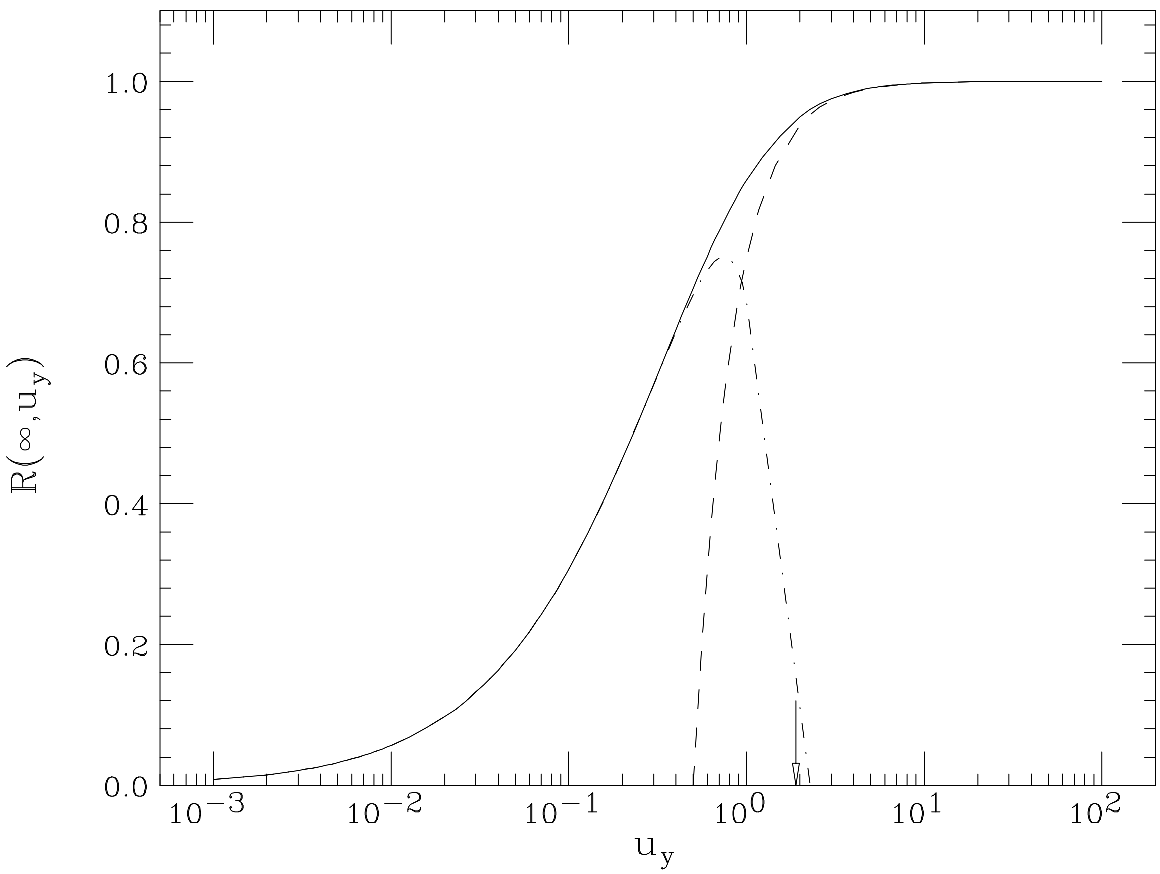


Figure 2: Luminosity reduction factor for flat beams. Eq. (2.12) (solid line) is plotted vs. $u_y$ along with its $u_y \to 0$ (dot-dash line) and $u_y \to \infty$ (dashed line) limits, Eqs. (2.13). The arrow corresponds to the APIARY 6.3-D design of the SLAC/LBL B-factory, showing a luminosity reduction of 5.5% relative to the nominal value.

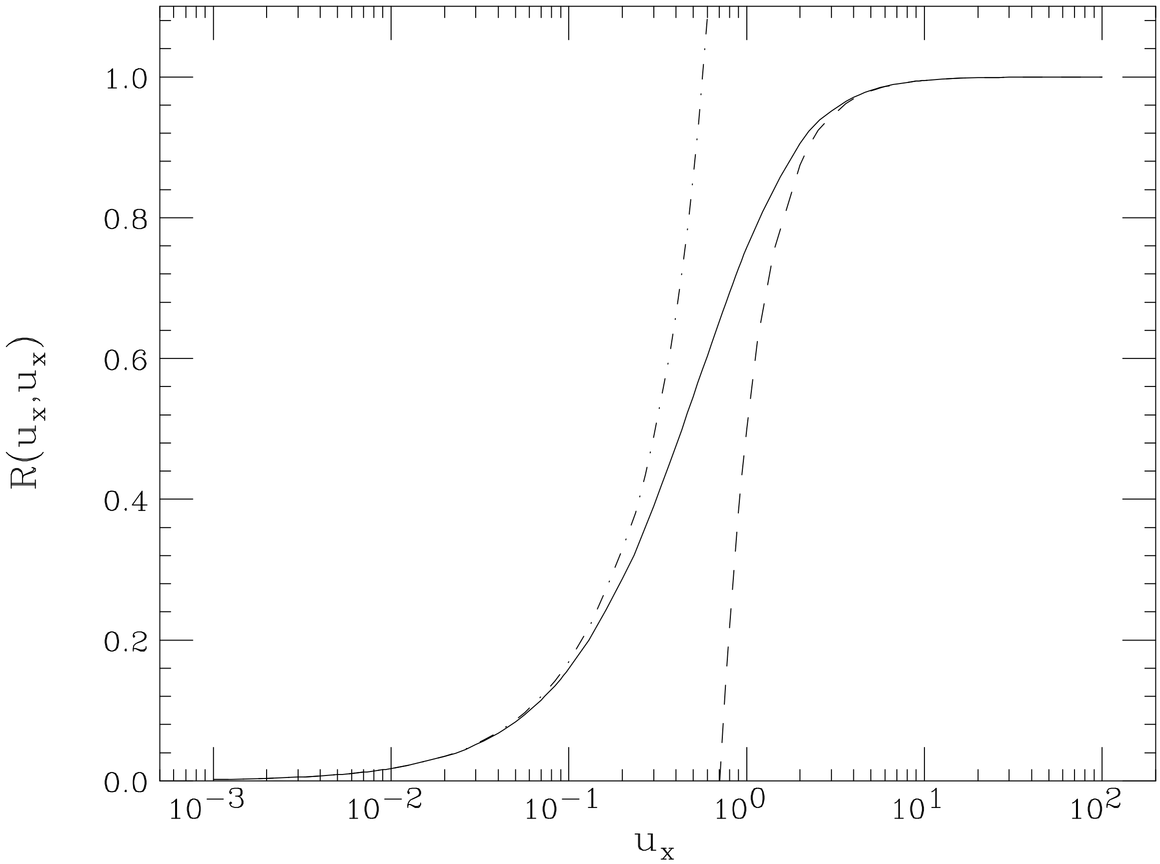


Figure 3: Luminosity reduction factor for round beams. Eq. (2.14) (solid line) is plotted vs. $u_x$ along with its $u_x \to 0$ (dot-dash line) and $u_x \to \infty$ (dashed line) limits, Eqs. (2.15).

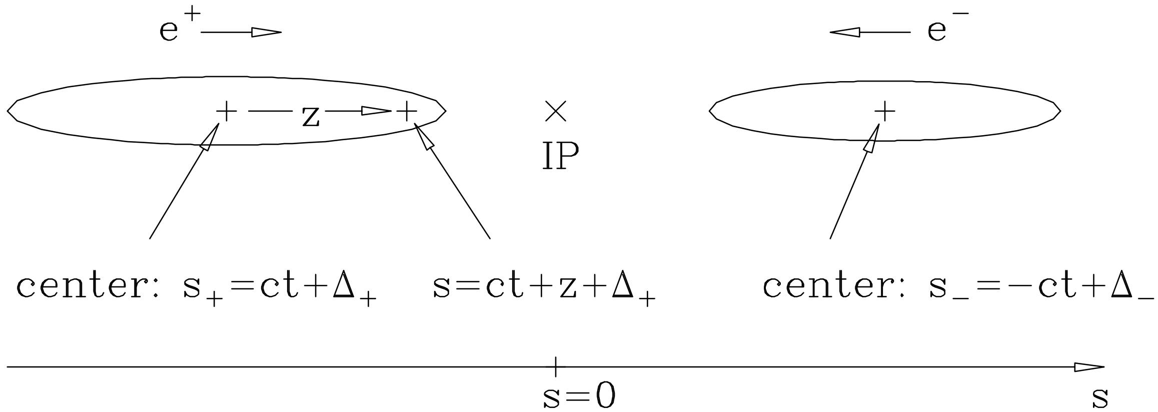


Figure 4: Sketch of the two bunches just before a collision. If the offsets $\Delta_+$ and $\Delta_-$ are 0, the bunch centers reach the interaction point (IP), $s = 0$, simultaneously at time $t = 0$. A particular positron at the head of the bunch is displaced by a distance $z$ from the center. If the offsets $\Delta_+$ and $\Delta_-$ are $\neq 0$, the bunches collide away from the IP at a distance $s_c = (\Delta_+ + \Delta_-)/2$ (see Sec. 5).

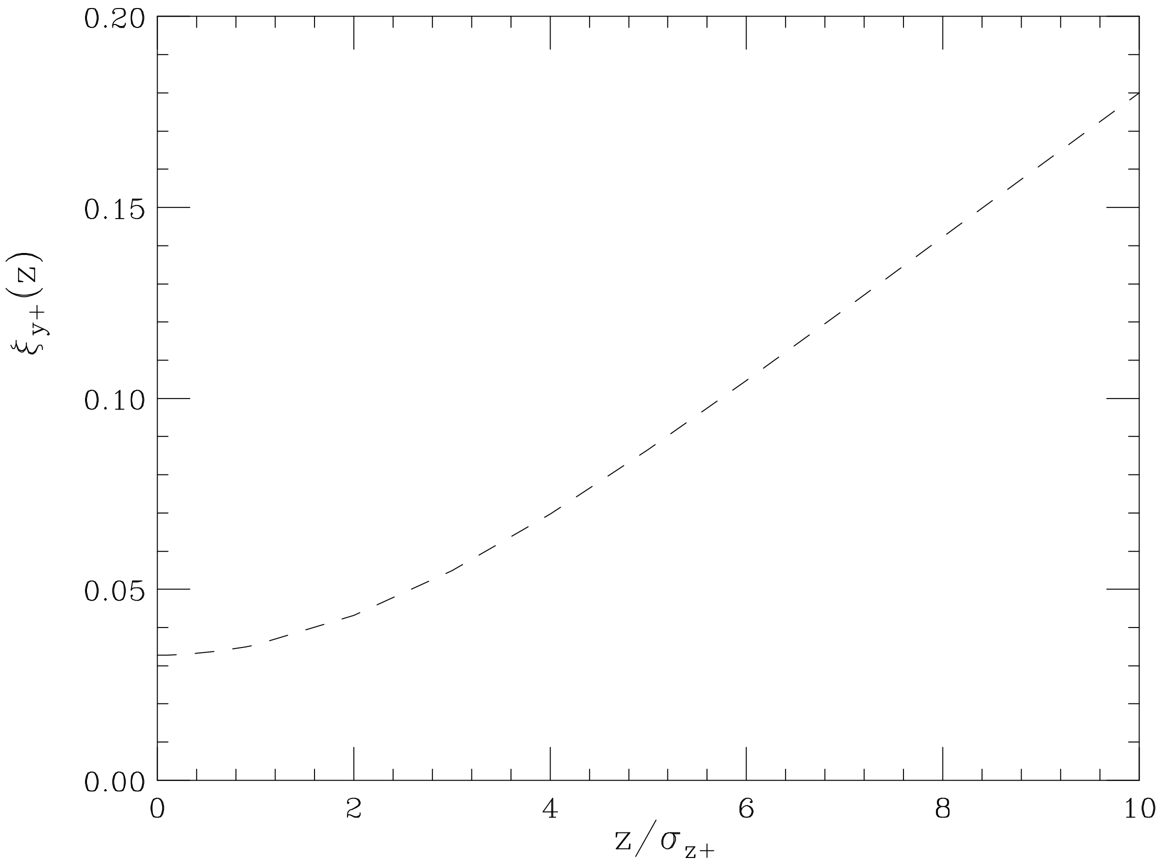


Figure 5: Vertical beam-beam parameter of the LER. The parameter $\xi_{y+}$ of a positron at a distance $z$ from the center of the bunch, obtained from Eq. (3.6), is plotted as a function of $z/\sigma_{z+}$. Nominal APIARY 6.3-D parameters are assumed ($\sigma_{z+} = 1$ cm).

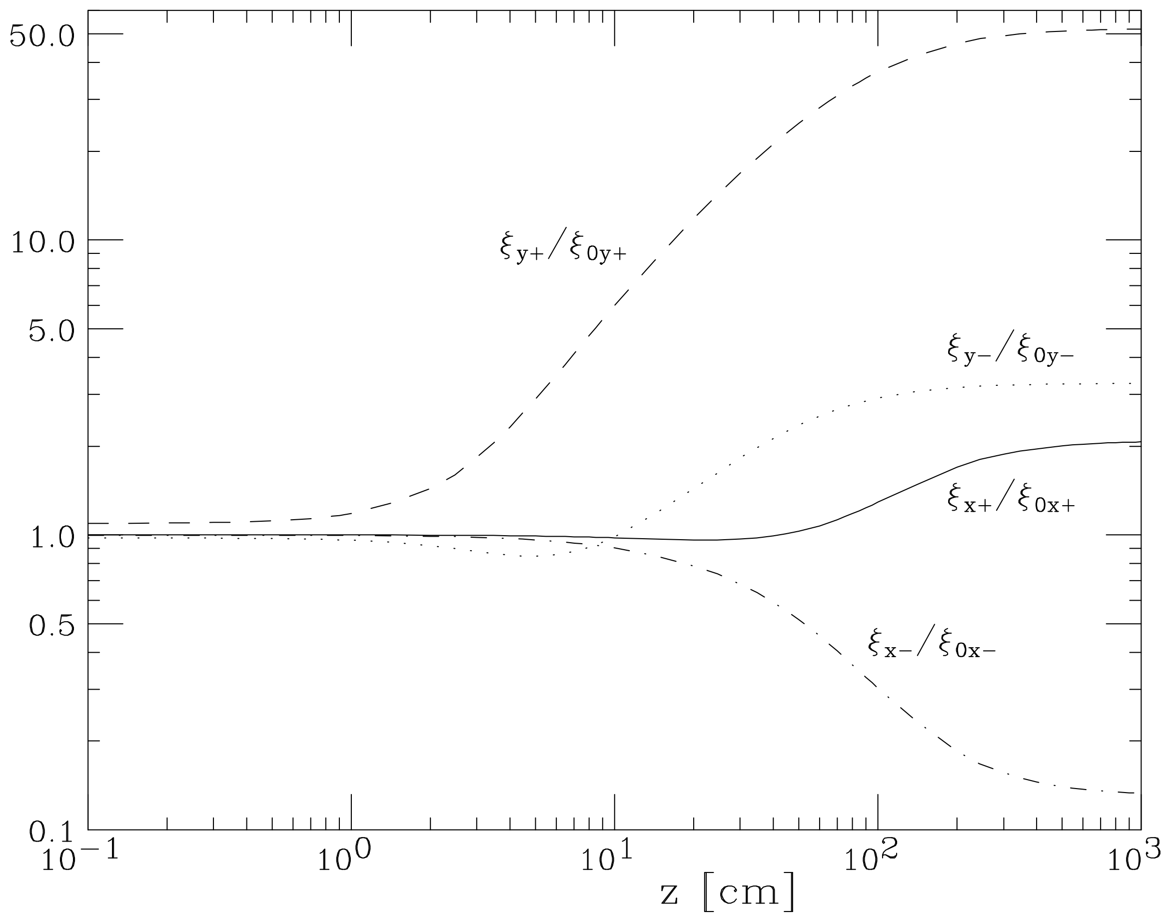


Figure 6: Beam-beam aggravating factors. The aggravating factors for a positron and an electron at a distance $z$ from the bunch center are obtained from Eq. (3.6), assuming nominal APIARY 6.3-D parameter values ($\sigma_{z+} = \sigma_{z-} = 1$ cm). The horizontal scale is extended up to an unphysically large value corresponding to $z = 10$ m in order to show the mathematical saturation property, Eq. (3.16).

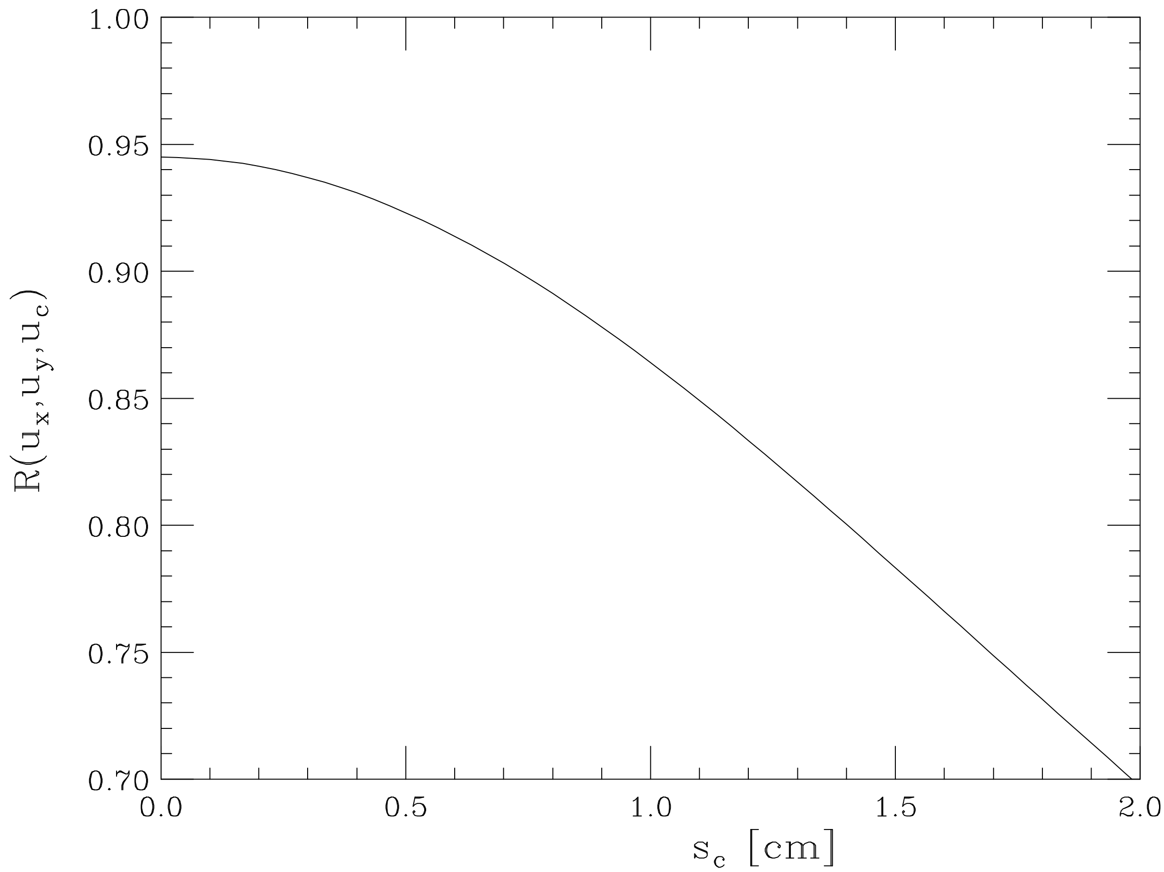


Figure 7: Luminosity reduction for off-IP collisions in APIARY 6.3-D. The centers of the bunches come together at a distance $s_c$ from the IP. The rms bunch lengths are $\sigma_{z+} = \sigma_{z-} = 1$ cm. The convention is that $s_c > 0$ means downstream of the positron beam, but the curve is symmetric about $s_c = 0$.

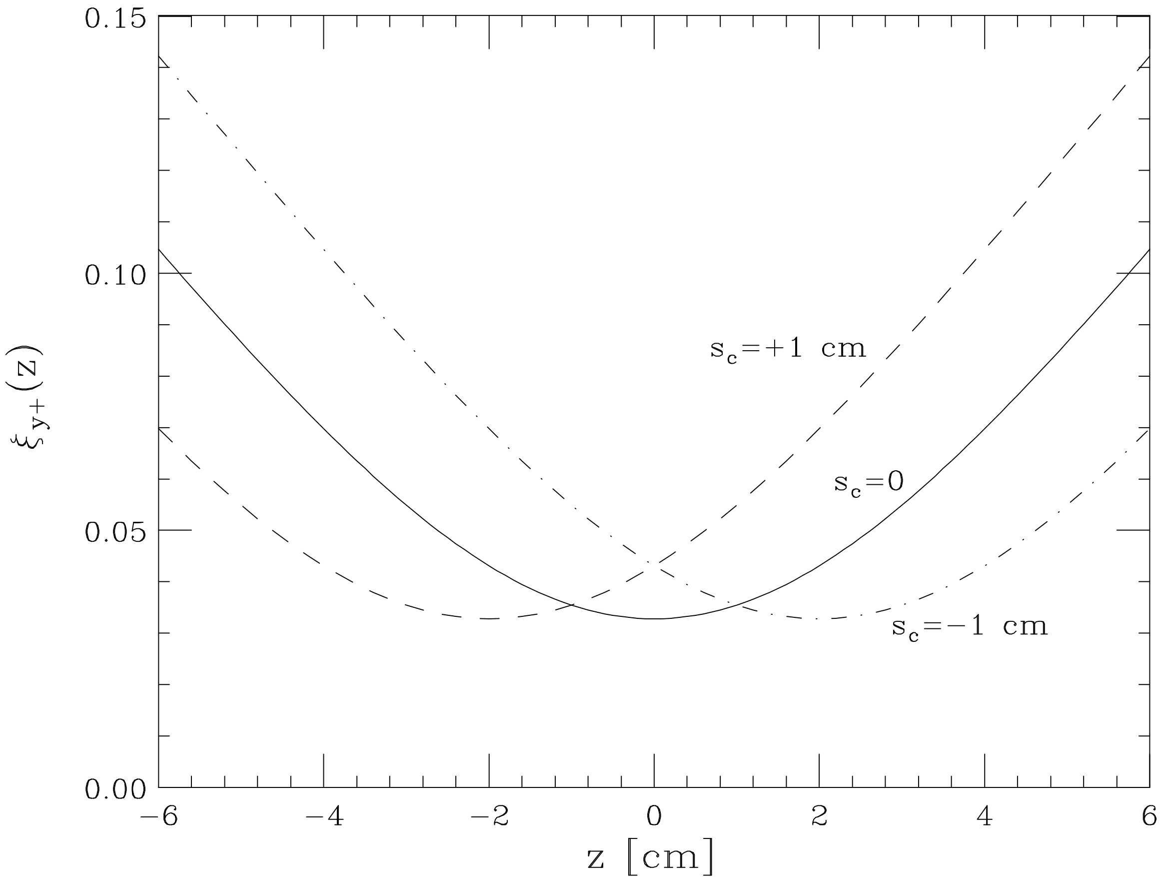


Figure 8: $\xi_{y+}(z)$ for off-IP collisions in APIARY 6.3-D. The vertical beam-beam parameter of a positron at a distance $z$ from the center of its own bunch. The head of the bunch is at $z > 0$, the tail at $z < 0$. The centers of the $e^+$ and $e^-$ bunches come together at a distance $s_c$ from the optical IP, where $s_c > 0$ means downstream of the positron bunch. The rms bunch lengths are $\sigma_{z+} = \sigma_{z-} = 1$ cm. For the electron bunch, the convention for $z$ is reversed: $z > 0$ represents the tail of the bunch, $z < 0$ the head.